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PERTURBATION THEORY OF NONLINEAR ELASTIC WAVE PROPAGATION*

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Abstract—In this article a perturbation method is given for obtaining uniformly valid approximations to the solution of problems of plane wave propagation in finite elasticity theory. Examples are given showing its application to problems of longitudinal and shear wave propagation, including an explicit numerical example concerning the propagation of a shear wave in a polyurethane foam rubber.

1. INTRODUCTION

IN a previous paper [1] some exact and general solutions were given for problems in the theory of propagation of plane waves of finite amplitude in elastic solids. Similar problems have recently been considered by Bland [2], Varley [3], John [4], Collins [5], and Howard [6]. The application of these results rests upon the integration of a first-order ordinary differential equation, the form of which depends on the material constitution. It is at this point that progress usually ceases if we consider the most general admissible materials. To proceed further it is customary to restrict attention to special materials that are usually presumed to model real materials accurately only for moderate deformation. In most applications it is desirable to obtain solutions for more general boundary conditions than the step functions considered in [1]. For the above reasons we see that, as a practical matter, what is really needed is an approximation to the solution that, while it may be valid only for moderate deformations, still illuminates the nonlinear behavior. For example, it should exhibit the same smoothness as the exact solution and should show how the waveform changes as the wave propagates. Such an approximation to the solution can be obtained with less knowledge of the constitution of the material than is required for the exact solution.

One obvious method of approximation is a perturbation analysis. The simplest such analysis, a straight-forward perturbation expansion, has been discussed by Fine and Shield [7] for general three-dimensional elastodynamic problems. Since this method fails to incorporate improved estimates of the wavespeed at each stage of the computation, it generates secular terms in the solution. Thus we find that while this method is quite easy to use and is applicable in much more general circumstances than the one to be proposed here it is deficient for the present purpose in that it gives its best results only for a rather short time interval, and then in a form which somewhat obscures the qualitative aspects of the solution. A similar difficulty frequently occurs in investigation of ordinary differential equations having periodic solutions but can be avoided there by applying, for example, Lindsted's method of correcting the frequency of oscillation.

A perturbation analysis is described in this paper which incorporates a corrected estimate for the wavespeed at each stage of the analysis, thus avoiding the appearance of

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secular terms in the solution and giving results which are uniformly valid and easily interpretable. This method is similar to that proposed by Lin [8] and Fox [9] in gasdynamical contexts, but has been modified somewhat to include shear wave problems. The question of convergence of the approximations is not discussed here, but is covered in [8] and [9] for the simpler cases discussed there.

Section 2 of this article is devoted to formulation of the field equations and boundary conditions. In Section 3 perturbation equations of order zero, one, and two, along with the associated boundary and initial conditions, are obtained. The general solution for this system of perturbation equations is given explicitly in Section 4. In Sections 5, 6, and 7 the general results of Section 4 are applied to two special problems involving the propagation of longitudinal and shear loading waves.

2. FIELD EQUATIONS AND BOUNDARY CONDITIONS

The governing equations of the theory express: (i) the mechanical principles of balance of linear momentum, angular momentum, and mass, and (ii) the mechanical constitution of the material. For reasons given in [1], equations expressing the thermodynamic principles of balance of energy and of entropy production are not used.

Let the material under study reside in an inertial space, the points of which are denoted by their coordinates x_i in a Cartesian reference frame x. Let the particles of the body be denoted by their place X_i in the x-space when the body is in its initial configuration.

The problem to be considered concerns an elastic material, homogeneous, isotropic, unstressed, and at rest in the initial state, and occupying the half-space $X_1 \geq 0$. In the initial state the displacement gradients are zero throughout the domain. For $t > 0$ new values are specified for the displacement gradients on $X_1 = 0$ and the resulting waves are investigated.

The motion described above may be written in the form

$$x_1 = X_1 + U_1(X_1, t), \qquad x_2 = X_2 + U_2(X_1, t), \qquad x_3 = X_3, \tag{2.1}$$

where $\mathbf{U}$ is the material form of the displacement vector measured with respect to the initial configuration, and where the invertibility of the motion (2.1) is taken as a mechanical principle. To simplify the formulae of the following sections we define the new quantities

$$P \equiv U_{1x_1}, \qquad Q \equiv U_{2x_1}, \qquad R \equiv U_{1_t}, \qquad S \equiv U_{2_t}. \tag{2.2}$$

The density, ρ, of the material in the deformed state can be expressed in terms of the (constant) density, ρ_0, of the material in the initial configuration by the formula

$$\rho = \rho_0/(1+P). \tag{2.3}$$

The quantity $1+P$, and thus the density in the deformed state, is finite and nonzero because of the postulated invertibility of the motion (2.1).

The fields encountered in the following work will be smooth enough that $\mathbf{U}(X_1, t)$ is continuous for $X_1, t > 0$. Indeed, all the first and second derivatives of $\mathbf{U}$ will be assumed to exist and to be continuous except at a finite number of distinct propagating singular surfaces.

The only type of singular surfaces considered in this article are acceleration waves. A surface Σ is said to be an acceleration wave if the fields in a neighborhood of this surface are smooth as above except that finite jumps occur in the second derivatives $\mathbf{U}_{X_1X_1}$, $\mathbf{U}_{X_1t}$, and $\mathbf{U}_{tt}$ across Σ.

For the material under consideration the stresses are given in terms of the displacement gradients by the relation

$$\mathbf{t} = h_{-1}\mathbf{c}^{-1} + h_0\mathbf{I} + h_1\mathbf{c} \tag{2.4}$$

where the response coefficients h_Γ depend on the scalar invariants of $\mathbf{c}^{-1}$, the inverse Cauchy deformation tensor, and are characteristic of the material. For simplicity we write

$$t_{11} = \sigma(P, Q), \qquad t_{12} = \tau(P, Q) \tag{2.5}$$

where σ and τ are functions having the form given by (2.4). It will be assumed in the following that σ and τ are continuously differentiable functions of P and Q. Examination of the function $\tau(P, Q)$ shows that $\tau(P, 0) = \tau_P(P, 0) = 0$ for all P and for all choices of the response functions. Various restrictions will be placed on the functions σ and τ and their derivatives in the following to ensure response of the material which seems plausible in various physical situations.

From the mechanical principles mentioned earlier, and the smoothness assumptions, we find that in regions where $\mathbf{U}(X, t)$ is twice continuously differentiable the fields are described by the equations

$$\begin{aligned} \rho_0 R_t - \sigma_{X_1} &= 0, & \rho_0 S_t - \tau_{X_1} &= 0, \\ P_t - R_{X_1} &= 0, & Q_t - S_{X_1} &= 0. \end{aligned} \tag{2.6}$$

Across an acceleration wave the field quantities P, Q, R and S are required to be continuous. Since it has been assumed that σ and τ are differentiable functions of the displacement gradients P and Q, (2.6) can be written in the form

$$\begin{aligned} \alpha P_X + \beta Q_X &= R_t, & \gamma P_X + \delta Q_X &= S_t, \\ R_X &= P_t, & S_X &= Q_t, \end{aligned} \tag{2.7}$$

where

$$\rho_0\alpha = \sigma_P(P, Q), \qquad \rho_0\beta = \sigma_Q(P, Q), \qquad \rho_0\gamma = \tau_P(P, Q), \qquad \rho_0\delta = \tau_Q(P, Q) \tag{2.8}$$

and where, for brevity, X has been written in place of X_1.

In order that disturbances of the elastic materials here considered propagate through the material as waves, conditions must be met by the coefficients in the system (2.7) which guarantee that it be of hyperbolic type. Formal computation of the characteristic wavespeeds gives them as solutions, a, of the equation

$$a^4 - (\alpha + \delta)a^2 + (\alpha\delta - \beta\gamma) = 0. \tag{2.9}$$

From this the four characteristic wavespeeds are found to be

$$\begin{aligned} a_4 &= -a_1 = \{\tfrac{1}{2}[\alpha + \delta + ((\alpha - \delta)^2 + 4\beta\gamma)^{\frac{1}{2}}]\}^{\frac{1}{2}}, \\ a_3 &= -a_2 = \{\tfrac{1}{2}[\alpha + \delta - ((\alpha - \delta)^2 + 4\beta\gamma)^{\frac{1}{2}}]\}^{\frac{1}{2}}, \end{aligned} \tag{2.10}$$

where we must require

$$(\alpha - \delta)^2 + 4\beta\gamma > 0, \qquad \alpha + \delta > [(\alpha - \delta)^2 + 4\beta\gamma]^{\frac{1}{2}} \tag{2.11}$$

in order that the wavespeeds be real. The first of these inequalities is automatically satisfied for hyperelastic materials, β and γ being equal in that case. Otherwise we take (2.11) as restrictions on the material constitution. As will be discussed later the wavespeeds satisfy the relations

$$a_1 < a_2 < 0 < a_3 < a_4. \tag{2.12}$$

The following limits are assumed to exist as $P, Q \to 0$:

$$\alpha \to (\lambda + 2\mu)/\rho_0, \qquad \beta, \gamma \to 0, \qquad \delta \to \mu/\rho_0 \tag{2.13}$$

where λ and μ are constants characteristic of the material. These prescriptions are enforced so that, for small deformations, the nonlinear theory used here reduces to the usual linear theory of elasticity. The constants λ and μ are the Lamé moduli of that theory. In the linear theory it is usually assumed that $3\lambda + 2\mu > 0$ and $\mu > 0$ in order that the strain energy function be positive definite. These assumptions are made here and imply

$$\lambda + \mu > 0. \tag{2.14}$$

Use of these results leads to the limiting values

$$a_3 \to \sqrt{(\mu/\rho_0)}, \qquad a_4 \to \sqrt{[(\lambda + 2\mu)/\rho_0]}, \tag{2.15}$$

as $P, Q \to 0$, for the wavespeeds (2.10). In the linear theory $(2.15)_1$ is the shear wavespeed and $(2.15)_2$ is the longitudinal wavespeed. These names are retained for a_3 and a_4 in the nonlinear theory although, as we will see, coupling effects make the names somewhat less meaningful. By (2.14), (2.15) and the assumed continuity of α, β, γ and δ as functions of P and Q,

$$a_3 < a_4 \tag{2.16}$$

for P and Q sufficiently small. It is assumed in this article that (2.16) is true for all relevant values of P and Q for all materials under consideration.

The initial and boundary conditions for the general problem described at the beginning of this section have the mathematical form

$$P(X, 0) = 0, \qquad Q(X, 0) = 0, \qquad R(X, 0) = 0, \qquad S(X, 0) = 0, \qquad X \geq 0,$$
$$P(0, t) = \varepsilon\Phi(t), \qquad Q(0, t) = \varepsilon\Psi(t), \tag{2.17}$$

with

$$\Phi(t) = \Psi(t) \equiv 0 \text{ for } t \leq 0, \tag{2.18}$$

where Φ and Ψ are given functions and $\varepsilon \ll 1$ is a dimensionless constant.

The basic idea behind the perturbation method to be proposed here is to incorporate improved estimates for the wavespeeds at each stage of the analysis. This is essentially equivalent to saying that we need improving estimates of the location in the (X, t)-plane of the characteristic base curves of (2.7). It turns out to be convenient to invert this process, taking a pair of the characteristics as coordinates and seeking improving estimates of the location of curves of constant X or constant t in the plane defined by these characteristics. Since we will be dealing with advancing waves it is appropriate to take the two families of advancing characteristics as coordinates. Thus we introduce new independent variables

s_1 and s_2 related to the physical variables X and t by means of the equations

$$X_{s_1} - a_4 t_{s_1} = 0, \qquad X_{s_2} - a_3 t_{s_2} = 0, \tag{2.19}$$

where a_3 and a_4, given by (2.10), are the two positive characteristic wavespeeds. The new variables constitute an admissible coordinate system so long as the wavespeeds a_3 and a_4 are distinct, finite, and nonzero. Upon transforming the field equations (2.7) to (s_1, s_2)-coordinates and adjoining (2.19) we obtain the system

$$\begin{aligned} t_{s_2}(\alpha P_{s_1} + \beta Q_{s_1} + a_3 R_{s_1}) - t_{s_1}(\alpha P_{s_2} + \beta Q_{s_2} + a_4 R_{s_2}) &= 0, \\ t_{s_2}(\gamma P_{s_1} + \delta Q_{s_1} + a_3 S_{s_1}) - t_{s_1}(\gamma P_{s_2} + \delta Q_{s_2} + a_4 S_{s_2}) &= 0, \\ t_{s_2}(R_{s_1} + a_3 P_{s_1}) - t_{s_1}(R_{s_2} + a_4 P_{s_2}) &= 0, \\ t_{s_2}(S_{s_1} + a_3 Q_{s_1}) - t_{s_1}(S_{s_2} + a_4 Q_{s_2}) &= 0, \\ X_{s_1} - a_4 t_{s_1} &= 0, \\ X_{s_2} - a_3 t_{s_2} &= 0. \end{aligned} \tag{2.20}$$

The problem of application of strain at the boundary of a half-space initially at rest and undeformed is set in terms of the field equations (2.20) and the initial and boundary conditions obtained by expressing (2.17) in a suitable form in the (s_1, s_2)-coordinates. In addition to this we must fix the parametrization of the characteristics by placing initial conditions on X and t. These conditions are taken as

$$X(s, s) = 0, \qquad a_0 t(s, s) = s, \tag{2.21}$$

where a_0 is some constant having the dimensions of a velocity. With this choice of parametrization the characteristics are labeled as shown in Fig. 1.

In consideration of (2.21), Equations (2.18) can be rewritten in terms of the (s_1, s_2)-coordinates as

$$P(s, s) = \varepsilon\Phi(s/a_0), \qquad Q(s, s) = \varepsilon\Psi(s/a_0). \tag{2.22}$$

Since the longitudinal wavefront is $s_2 = 0$ and the shear wavefront is $s_1 = 0$ equations $(2.17)_{1\text{--}4}$ can be replaced by

$$P(s_1, 0) = Q(0, s_2) = 0, \qquad R(s_1, 0) = S(0, s_2) = 0. \tag{2.23}$$

3. PERTURBATION EXPANSIONS

We assume that the dependent variables in (2.20) are representable by the perturbation series

$$\begin{aligned} P &= \varepsilon P_1 + \varepsilon^2 P_2 + \ldots, & Q &= \varepsilon Q_1 + \varepsilon^2 Q_2 + \ldots, \\ R &= \varepsilon R_1 + \varepsilon^2 R_2 + \ldots, & S &= \varepsilon S_1 + \varepsilon^2 S_2 + \ldots, \\ X &= X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \ldots, & t &= t_0 + \varepsilon t_1 + \varepsilon^2 t_2 + \ldots, \end{aligned} \tag{3.1}$$

where the perturbation quantities are regarded as functions of s_1 and s_2.

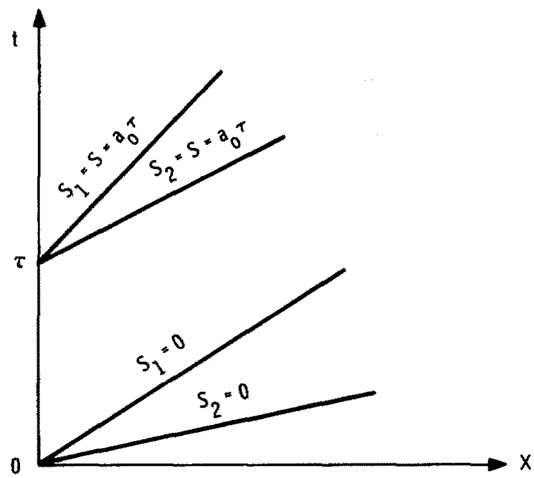

FIG. 1

Expansion of (2.8) gives

$$\begin{aligned}
\alpha &= \alpha_{00} + \varepsilon\alpha_{10}P_1 + \varepsilon^2(\alpha_{10}P_2 + \alpha_{02}Q_1{}^2 + \alpha_{20}P_1{}^2) + \ldots, \\
\beta &= \varepsilon\beta_{01}Q_1 + \varepsilon^2(\beta_{01}Q_2 + 2\alpha_{02}P_1Q_1) + \ldots, \\
\gamma &= \varepsilon\delta_{10}Q_1 + \varepsilon^2(\delta_{10}Q_2 + 2\delta_{20}P_1Q_1) + \ldots, \\
\delta &= \delta_{00} + \varepsilon\delta_{10}P_1 + \varepsilon^2(\delta_{10}P_2 + \delta_{20}P_1{}^2 + \delta_{02}Q_1{}^2) + \ldots,
\end{aligned} \tag{3.2}$$

where

$$\alpha_{00} = (\lambda + 2\mu)/\rho_0, \qquad \delta_{00} = \mu/\rho_0$$

are the squares of the linear wavespeeds and the remaining coefficients are second and third order elastic constants. Using these results we obtain the expansions

$$\begin{aligned}
a_3 &= \sqrt{\delta_{00}} + \varepsilon a_{31}P_1 + \varepsilon^2(a_{31}P_2 + a_{32}Q_1{}^2 + a_{33}P_1{}^2) + \ldots, \\
a_4 &= \sqrt{\alpha_{00}} + \varepsilon a_{41}P_1 + \varepsilon^2(a_{41}P_2 + a_{42}Q_1{}^2 + a_{43}P_1{}^2) + \ldots,
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
a_{31} &= \frac{\delta_{10}}{2\sqrt{\delta_{00}}}, \quad a_{32} = \frac{1}{2\sqrt{\delta_{00}}}\left(\delta_{02} - \frac{\beta_{01}\delta_{10}}{\alpha_{00} - \delta_{00}}\right), \quad a_{33} = \frac{1}{2\sqrt{\delta_{00}}}\left(\delta_{20} - \frac{\delta_{10}{}^2}{4\delta_{00}}\right), \\
a_{41} &= \frac{\alpha_{10}}{2\sqrt{\alpha_{00}}}, \quad a_{42} = \frac{1}{2\sqrt{\alpha_{00}}}\left(\alpha_{02} + \frac{\beta_{01}\delta_{10}}{\alpha_{00} - \delta_{00}}\right), \quad a_{43} = \frac{1}{2\sqrt{\alpha_{00}}}\left(\alpha_{20} - \frac{\alpha_{10}{}^2}{4\alpha_{00}}\right),
\end{aligned} \tag{3.4}$$

for the characteristic wavespeeds given by (2.10).

Substitution of (3.1)–(3.3) into (2.20) and equating to zero the coefficients of the zeroth, first, and second powers of ε gives the perturbation equations:

$$\varepsilon^0: (X_0-\sqrt{\alpha_{00}}\,t_0)_{s_1} = 0, \qquad (X_0-\sqrt{\delta_{00}}\,t_0)_{s_2} = 0, \tag{3.5}$$

$$\begin{aligned}
\varepsilon_1: &\ t_{0_{s_2}}(\alpha_{00}P_1+\sqrt{\delta_{00}}\,R_1)_{s_1} - t_{0_{s_1}}(\alpha_{00}P_1+\sqrt{\alpha_{00}}\,R_1)_{s_2} = 0,\\
&\ t_{0_{s_2}}(\delta_{00}Q_1+\sqrt{\delta_{00}}\,S_1)_{s_1} - t_{0_{s_1}}(\delta_{00}Q_1+\sqrt{\alpha_{00}}\,S_1)_{s_2} = 0,\\
&\ t_{0_{s_2}}(R_1+\sqrt{\delta_{00}}\,P_1)_{s_1} - t_{0_{s_1}}(R_1+\sqrt{\alpha_{00}}\,P_1)_{s_2} = 0,\\
&\ t_{0_{s_2}}(S_1+\sqrt{\delta_{00}}\,Q_1)_{s_1} - t_{0_{s_1}}(S_1+\sqrt{\alpha_{00}}\,Q_1)_{s_2} = 0,\\
&\ (X_1-\sqrt{\alpha_{00}}\,t_1)_{s_1} = a_{41}P_1 t_{0_{s_1}},\\
&\ (X_1-\sqrt{\delta_{00}}\,t_1)_{s_2} = a_{31}P_1 t_{0_{s_2}},
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
\varepsilon^2: &\ t_{0_{s_2}}(\alpha_{00}P_2+\sqrt{\delta_{00}}\,R_2)_{s_1} - t_{0_{s_1}}(\alpha_{00}P_2+\sqrt{\alpha_{00}}\,R_2)_{s_2}\\
&\quad = t_{1_{s_1}}(\alpha_{00}P_1+\sqrt{\alpha_{00}}\,R_1)_{s_2} - t_{1_{s_2}}(\alpha_{00}P_1+\sqrt{\delta_{00}}\,R_1)_{s_1}\\
&\quad\quad + t_{0_{s_1}}(\alpha_{10}P_1P_{1_{s_2}}+\beta_{01}Q_1Q_{1_{s_2}}+a_{41}P_1R_{1_{s_2}})\\
&\quad\quad - t_{0_{s_2}}(\alpha_{10}P_1P_{1_{s_1}}+\beta_{01}Q_1Q_{1_{s_1}}+a_{31}P_1R_{1_{s_1}}),\\
&\ t_{0_{s_2}}(\delta_{00}Q_2+\sqrt{\delta_{00}}\,S_2)_{s_1} - t_{0_{s_1}}(\delta_{00}Q_2+\sqrt{\alpha_{00}}\,S_2)_{s_2}\\
&\quad = t_{1_{s_1}}(\delta_{00}Q_1+\sqrt{\alpha_{00}}\,S_1)_{s_2} - t_{1_{s_2}}(\delta_{00}Q_1+\sqrt{\delta_{00}}\,S_1)_{s_1}\\
&\quad\quad + t_{0_{s_1}}(\delta_{10}(P_1Q_1)_{s_2}+a_{41}P_1S_{1_{s_2}})\\
&\quad\quad - t_{0_{s_2}}(\delta_{10}(P_1Q_1)_{s_1}+a_{31}P_1S_{1_{s_1}}),\\
&\ t_{0_{s_2}}(R_2+\sqrt{\delta_{00}}\,P_2)_{s_1} - t_{0_{s_1}}(R_2+\sqrt{\alpha_{00}}\,P\ )_{s_2}\\
&\quad = t_{1_{s_1}}(R_1+\sqrt{\alpha_{00}}\,P_1)_{s_2} - t_{1_{s_2}}(R_1+\sqrt{\delta_{00}}\,P_1)_{s_1}\\
&\quad\quad + \tfrac{1}{2}t_{0_{s_1}}a_{41}(P_1{}^2)_{s_2} - \tfrac{1}{2}t_{0_{s_2}}a_{31}(P_1{}^2)_{s_1}\\
&\ t_{0_{s_2}}(S_2+\sqrt{\delta_{00}}\,Q_2)_{s_1} - t_{0_{s_1}}(S_2+\sqrt{\alpha_{00}}\,Q_2)_{s_2}\\
&\quad = t_{1_{s_1}}(S_1+\sqrt{\alpha_{00}}\,Q_1)_{s_2} - t_{1_{s_2}}(S_1+\sqrt{\delta_{00}}\,Q_1)_{s_1}\\
&\quad\quad - t_{0_{s_2}}a_{31}P_1Q_{1_{s_1}} + t_{0_{s_1}}a_{41}P_1Q_{1_{s_2}},\\
&\ (X_2-\sqrt{\alpha_{00}}\,t_2)_{s_1} = a_{41}P_1t_{1_{s_1}} + t_{0_{s_1}}(a_{41}P_2+a_{42}Q_1{}^2+a_{43}P_1{}^2),\\
&\ (X_2-\sqrt{\delta_{00}}\,t_2)_{s_2} = a_{31}P_1t_{1_{s_2}} + t_{0_{s_2}}(a_{31}P_2+a_{32}Q_1{}^2+a_{33}P_1{}^2).
\end{aligned} \tag{3.7}$$

By perturbation of (2.21)–(2.23) we find the boundary and initial conditions for the above problems to be:

$$\varepsilon^0: X_0(s,s) = 0, \qquad t_0(s,s) = s/a_0, \tag{3.8}$$

$$\begin{aligned}
\varepsilon^1: &\ P_1(s,s) = \Phi(s/a_0), \qquad Q_1(s,s) = \Psi(s/a_0),\\
&\ P_1(s_1,0) = Q_1(0,s_2) = 0, \qquad R_1(s_1,0) = S_1(0,s_2) = 0,\\
&\ X_1(s,s) = 0, \qquad t_1(s,s) = 0,
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
\varepsilon^2: &\ P_2(s,s) = 0, \qquad Q_2(s,s) = 0, \qquad X_2(s,s) = 0, \qquad t_2(s,s) = 0,\\
&\ P_2(s_1,0) = Q_2(0,s_2) = 0, \qquad R_2(s_1,0) = S_2(0,s_2) = 0.
\end{aligned} \tag{3.10}$$

4. SOLUTION OF THE PERTURBATION EQUATIONS

The finding of solutions to (3.5)–(3.7) satisfying (3.8)–(3.10) is a fairly routine matter that we need treat only briefly. The functions satisfying (3.5) and (3.8) are easily seen to be

$$t_0(s_1, s_2) = -\frac{\sqrt{\delta_{00}}\, s_1 - \sqrt{\alpha_{00}}\, s_2}{a_0(\sqrt{\alpha_{00}} - \sqrt{\delta_{00}})}, \qquad X_0(s_1, s_2) = -\frac{\sqrt{\alpha_{00}}\sqrt{\delta_{00}}}{a_0(\sqrt{\alpha_{00}} - \sqrt{\delta_{00}})}(s_1 - s_2). \tag{4.1}$$

Using this result we find the solution of $(3.6)_{1-4}$ subject to (3.7) to be

$$\begin{aligned} P_1(s_1, s_2) &= \Phi(s_2/a_0), \qquad & R_1(s_1, s_2) &= -\sqrt{\alpha_{00}}\,\Phi(s_2/a_0), \\ Q_1(s_1, s_2) &= \Psi(s_1/a_0), \qquad & S_1(s_1, s_2) &= -\sqrt{\delta_{00}}\,\Psi(s_1/a_0). \end{aligned} \tag{4.2}$$

The results (4.1) and (4.2) are the solutions associated with the linear theory of elasticity. Results obtained beyond this point are contributions associated with the nonlinear features of the problem and the phenomena predicted do not occur in the linear theory. After substituting the above results into $(3.6)_{5,6}$ we can solve these equations to obtain

$$\begin{aligned} X_1(s_1, s_2) &= f_1 f_8 \sqrt{\delta_{00}}\left(\frac{s_1 - s_2}{a_0}\right)\Phi\left(\frac{s_2}{a_0}\right) + f_4 f_8 \sqrt{\alpha_{00}} \int_{s_1/a_0}^{s_2/a_0} \Phi(\xi)\,\mathrm{d}\xi, \\ t_1(s_1, s_2) &= f_1 f_8 \frac{s_1 - s_2}{a_0}\Phi\left(\frac{s_2}{a_0}\right) + f_4 f_8 \int_{s_1/a_0}^{s_2/a_0} \Phi(\xi)\,\mathrm{d}\xi, \end{aligned} \tag{4.3}$$

where

$$f_1 = \frac{a_{41}\sqrt{\delta_{00}}}{(\sqrt{\alpha_{00}} - \sqrt{\delta_{00}})}, \qquad f_4 = \frac{a_{31}\sqrt{\alpha_{00}}}{(\sqrt{\alpha_{00}} - \sqrt{\delta_{00}})}, \qquad f_8 = \frac{1}{\sqrt{\alpha_{00}} - \sqrt{\delta_{00}}}.$$

With this result we have completed the solution of the ε^1-equations. The functions $X(s_1, s_2)$ and $t(s_1, s_2)$ are now estimated to be

$$X = X_0 + \varepsilon X_1 + O(\varepsilon^2), \qquad t = t_0 + \varepsilon t_1 + O(\varepsilon^2), \tag{4.4}$$

where X_0, t_0, X_1, and t_1 are functions of s_1 and s_2 given by (4.1) and (4.3). To see that the slopes of the characteristics have now been corrected to agree with first-order wavespeeds we evaluate $(2.19)_1$. Using (4.4) we have

$$a_4 = \frac{X_{0s_1} + \varepsilon X_{1s_1} + \cdots}{t_{0_{s_1}} + \varepsilon t_{1_{s_1}} + \cdots} = \frac{X_{0_{s_1}}}{t_{0_{s_1}}} + \varepsilon \frac{1}{t_{0_{s_1}}}\left(X_{1_{s_1}} - \frac{X_{0_{s_1}}}{t_{0_{s_1}}} t_{1_{s_1}}\right) + O(\varepsilon^2)$$

which, by (4.1) and (4.3), may be written

$$a_4 = \sqrt{\alpha_{00}} + \varepsilon a_{41}\Phi(s_2/a_0) + O(\varepsilon^2).$$

We see that this is in agreement with the result obtained by substitution of P_1 into $(3.3)_1$.

It remains now to find solutions of the second order equations (3.7) consistent with the conditions (3.10). This time the computation is longer but still routine. The result is

$$
\begin{aligned}
P_2 &= \frac{f_7}{f_8}\left[\Psi^2\left(\frac{s_2}{a_0}\right) - \Psi^2\left(\frac{s_1}{a_0}\right)\right],\\
R_2 &= \frac{f_7}{f_8}\left[\sqrt{\delta_{00}}\,\Psi^2\left(\frac{s_1}{a_0}\right) - \sqrt{\alpha_{00}}\,\Psi^2\left(\frac{s_2}{a_0}\right)\right] - \tfrac{1}{2}a_{41}\Phi^2\left(\frac{s_2}{a_0}\right),\\
Q_2 &= \frac{1}{2K_4}\int_{-(K_4 s_1 - s_2)/a_0}^{(K_4 s_1 + s_2)/a_0} \left\{K_1\Psi'\left(\frac{\alpha + (K_4 s_1 - s_2)/a_0}{2K_4}\right)\Phi\left(\frac{\alpha - (K_4 s_1 - s_2)/a_0}{2}\right)\right.\\
&\quad \left. + K_2\Psi\left(\frac{\alpha + (K_4 s_1 - s_2)/a_0}{2K_4}\right)\Phi'\left(\frac{\alpha - (K_4 s_1 - s_2)/a_0}{2}\right)\right\} \mathrm{d}\alpha\\
&\quad + K_3 c(s_1/a_0),\\
S_2 &= \sqrt{\delta_{00}}\, Q_2 - \delta_{10} f_8[\Psi(s_1/a_0)\Phi(s_2/a_0)] - c(s_1/a_0),
\end{aligned}
\tag{4.5}
$$

where

$$
\begin{aligned}
K_1 &= a_{31}\left(\frac{3\sqrt{\delta_{00}} - \sqrt{\alpha_{00}}}{2\sqrt{\delta_{00}}\,(\sqrt{\alpha_{00}} - \sqrt{\delta_{00}})}\right), \qquad K_2 = \frac{\delta_{10}}{2\sqrt{a_{00}}\,(\sqrt{\alpha_{00}} - \sqrt{\delta_{00}})},\\
K_3 &= \frac{1}{2\sqrt{\delta_{00}}}, \qquad K_4 = \frac{\sqrt{\alpha_{00}} + \sqrt{\delta_{00}}}{2\sqrt{\alpha_{00}}},\\
c(x) &= \frac{1}{2K_3K_4}\int_{(K_4+1)x}^{-(K_4-1)x}\left\{K_1\Psi'\left(\frac{\alpha + (K_4 - 1)x}{2K_4}\right)\Phi\left(\frac{\alpha - (K_4 - 1)x}{2}\right)\right.\\
&\quad \left. + K_2\Psi\left(\frac{\alpha + (K_4 - 1)x}{2K_4}\right)\Phi'\left(\frac{\alpha - (K_4 - 1)x}{2}\right)\right\} \mathrm{d}\alpha.
\end{aligned}
\tag{4.6}
$$

With these results we proceed to compute the second-order corrections to X and t. We find that

$$
\begin{aligned}
X_2(s_1, s_2) &= \left[c_2\sqrt{\alpha_{00}}\,\Psi^2\left(\frac{s_1}{a_0}\right) + c_3\sqrt{\delta_{00}}\,\Psi^2\left(\frac{s_2}{a_0}\right) + c_7\Phi^2\left(\frac{s_2}{a_0}\right)\right]\left(\frac{s_1}{a_0} - \frac{s_2}{a_0}\right)\\
&\quad + \int_{s_1/a_0}^{s_2/a_0}\left[c_6\sqrt{\alpha_{00}}\,\Phi^2(\xi) + c_8\Psi^2(\xi) + c_5\sqrt{\delta_{00}}\,\Phi\left(\frac{s_2}{a_0}\right)\Phi(\xi)\right]\mathrm{d}\xi,\\
t_2(s_1, s_2) &= \left[c_1\Phi^2\left(\frac{s_2}{a_0}\right) + c_2\Psi^2\left(\frac{s_1}{a_0}\right) + c_3\Psi^2\left(\frac{s_2}{a_0}\right)\right]\left(\frac{s_1}{a_0} - \frac{s_2}{a_0}\right)\\
&\quad + \int_{s_1/a_0}^{s_2/a_0}\left[c_4\Psi^2(\xi) + c_6\Phi^2(\xi) + c_5\Phi\left(\frac{s_2}{a_0}\right)\Phi(\xi)\right]\mathrm{d}\xi,
\end{aligned}
\tag{4.7}
$$

where

$$
\begin{aligned}
c_1 &= f_1 f_8\left(\frac{f_4}{2\sqrt{\alpha_{00}}}-\frac{f_1}{\sqrt{\delta_{00}}}\right)+f_3 f_8, \qquad c_2 = f_4 f_7 - f_5 f_8,\\
c_3 &= f_1 f_7, \qquad c_4 = (f_1+f_4)f_7 - f_2 f_8, \qquad c_5 = -f_1 f_4 f_8/\sqrt{\delta_{00}},\\
c_6 &= f_8\left(f_6+\frac{f_4^{\,2}}{\sqrt{\alpha_{00}}}-\frac{f_1 f_4}{2\sqrt{\alpha_{00}}}\right), \qquad c_7 = f_8(f_3\sqrt{\delta_{00}}-f_1^{\,2}+\tfrac{1}{2}f_1 f_4),\\
c_8 &= \sqrt{\delta_{00}}c_4 + f_4 f_7/f_8,
\end{aligned}
\tag{4.8}
$$

with f_1, f_4, and f_8 as in (4.3) and with

$$
\begin{aligned}
f_2 &= f_8 a_{42}\sqrt{\delta_{00}}, \qquad f_3 = f_8 a_{43}\sqrt{\delta_{00}},\\
f_5 &= f_8 a_{32}\sqrt{\alpha_{00}}, \qquad f_6 = f_8 a_{33}\sqrt{\alpha_{00}}, \qquad f_7 = \frac{\beta_{01} f_8}{2(\alpha_{00}-\delta_{00})}.
\end{aligned}
\tag{4.9}
$$

To obtain the whole of the second-order solution we simply substitute the results of this section into (3.1).

We conceed the forbidding appearance of the formulae (4.5)–(4.7). One of the main reasons for the complexity is that the expressions are valid for all X and t. Since the solution for a hyperbolic problem can be very different in regions separated by the wavefronts, we may expect to find much simpler expressions valid in restricted domains of the (X, t)-plane. It was shown in [1] how the solution to the centered simple wave problem consists of three regions of uniform motion separated by a longitudinal and a shear wave. This property of the solution is implicit in the representation of this section. It is made explicit in the following sections where specific choices of Φ and Ψ are made. Having made a choice of Φ and Ψ it is easy to disentangle the complicated functions of (4.5)–(4.7) and to see the qualitative features of the solution quite clearly. Simple formulae are given from which one may easily obtain numerical results.

We have now determined

$$
\begin{gathered}
U_{1_X}(s_1, s_2), \qquad U_{1_t}(s_1, s_2), \qquad U_{2_X}(s_1, s_2), \qquad U_{2_t}(s_1, s_2),\\
X(s_1, s_2), \qquad t(s_1, s_2),
\end{gathered}
\tag{4.10}
$$

and through them the stresses, to a certain approximation, throughout the domain. If the latter two of these expressions be inverted to give $s_1 = s_1(X, t)$ and $s_2 = s_2(X, t)$ and if these results be substituted into $(4.10)_{1-4}$ the displacement gradients are then obtained for a given particle and time. The second and fourth of these quantities are the components of the particle velocity of the particle $\mathbf{X}$ at the time t. The first and third quantities are the displacement gradients as measured by a device stationed at the particle $\mathbf{X}$.

In some applications what is most needed is the function $\mathbf{u}(\mathbf{x}, t)$ giving the displacement in the spatial (laboratory) coordinates. In the following paragraphs it is shown how to obtain this displacement. Section 7 is devoted to a concrete numerical example of the calculation outlined.

From (4.10) we can write

$$U_{1_{s_1}} = U_{1_X}X_{s_1} + U_{1_t}t_{s_1} = PX_{s_1} + Rt_{s_1},$$
$$U_{1_{s_2}} = U_{1_X}X_{s_2} + U_{1_t}t_{s_2} = PX_{s_2} + Rt_{s_2}, \tag{4.11}$$
$$U_{2_{s_1}} = QX_{s_1} + St_{s_1}, \qquad U_{2_{s_2}} = QX_{s_2} + St_{s_2}.$$

Since P, Q, R, S, X, and t have already been determined (to second order) as functions of s_1 and s_2 we can compute

$$U_1(s_1, s_2) = \int (PX_{s_1} + Rt_{s_1})\,ds_1 + g_1(s_2)$$
$$= \int (PX_{s_2} + Qt_{s_2})\,ds_2 + f_1(s_1), \tag{4.12}$$
$$U_2(s_1, s_2) = \int (QX_{s_1} + St_{s_1})\,ds_1 + g_2(s_2)$$
$$= \int (QX_{s_2} + St_{s_2})\,ds_2 + f_2(s_1).$$

Substituting (3.1) into (4.12) we obtain

$$U_1(s_1, s_2) = \varepsilon \int (P_1 X_{0_{s_1}} + R_1 t_{0_{s_1}})\,ds_1 + \varepsilon^2 \int (P_1 X_{1_{s_1}} + P_2 X_{0_{s_1}}$$
$$+ R_1 t_{1_{s_1}} + R_2 t_{0_{s_1}})\,ds_1 + g_1(s_2) + O(\varepsilon^3),$$

and (4.13)

$$U_1(s_1, s_2) = \varepsilon \int (P_1 X_{0_{s_2}} + R_1 t_{0_{s_2}})\,ds_2 + \varepsilon^2 \int (P_1 X_{1_{s_2}} + P_2 X_{0_{s_2}}$$
$$+ R_1 t_{1_{s_2}} + R_2 t_{0_{s_2}})\,ds_2 + f_1(s_1) + O(\varepsilon^3).$$

Similarly

$$U_2(s_1, s_2) = \varepsilon \int (Q_1 X_{0_{s_1}} + S_1 t_{0_{s_1}})\,ds_1 + \varepsilon^2 \int (Q_1 X_{1_{s_1}} + Q_2 X_{0_{s_1}}$$
$$+ S_1 t_{1_{s_1}} + S_2 t_{0_{s_1}})\,ds_1 + g_2(s_2) + O(\varepsilon^3),$$

and (4.14)

$$U_2(s_1, s_2) = \varepsilon \int (Q_1 X_{0_{s_2}} + S_1 t_{0_{s_2}})\,ds_2 + \varepsilon^2 \int (Q_1 X_{1_{s_2}} + Q_2 X_{0_{s_2}}$$
$$+ S_1 t_{1_{s_2}} + S_2 t_{0_{s_2}})\,ds_2 + f_2(s_1) + O(\varepsilon^3).$$

Where the functions f_i, g_i are to be chosen so that $\mathbf{U}$ is continuous across the wavefronts and so that the two solutions in (4.12) and in (4.13) are in agreement.

From our previous results we have an expression for $t(s_1, s_2)$. Since $\mathbf{x} = \mathbf{X} + \mathbf{U}$ we can determine the function $\mathbf{x}(s_1, s_2)$ from the solutions obtained. Then, if we pick values for

s_1, s_2 we can determine the associated values for $\mathbf{x}, t, \mathbf{U}$. Since $\mathbf{u} = -\mathbf{U}$ we then have the associated values $(\mathbf{x}, t, \mathbf{u})$. Suppose we want to know the time dependence of $\mathbf{u}$ for some particular choice of $\mathbf{x}$, i.e. some particular laboratory station. We take

$$\mathbf{x} = \mathbf{X}(s_1, s_2)+\mathbf{U}(s_1, s_2),$$

which, for the given value $\mathbf{x}_0$ of $\mathbf{x}$, determines a curve through the (s_1, s_2)-space corresponding to the given place $\mathbf{x}_0$ at successive times. Suppose (4.14) is solved for s_1 to give

$$s_1 = \hat{s}_1(s_2; \mathbf{x}_0).$$

Substituting this into $t = t(s_1, s_2)$ we have

$$t = t(\hat{s}_1(s_2; x_0), s_2) = t(s_2; \mathbf{x}_0). \tag{4.15}$$

Now all we have to do is pick a succession of values of $s_2: s_2^{(1)}, s_2^{(2)}, \ldots, s_2^{(k)}$, compute the corresponding times from (4.15) and the displacements from

$$\mathbf{u}^{(k)} = -\mathbf{U}(\hat{s}_1(s_2^{(k)}; \mathbf{x}_0), s_2^{(k)}). \tag{4.16}$$

With sets of values from (4.15) and (4.16) we can plot the function $u = u(t; x_0)$ which is the desired result.

5. EXAMPLE 1: LONGITUDINAL WAVE

As a simple example of the foregoing results we consider the case

$$\Psi = 0, \qquad \varepsilon = P_0, \qquad a_0 = \sqrt{\alpha_{00}}, \tag{5.1}$$

corresponding to the application of forces normal to the boundary. Here, as always, we have t_0 and X_0 the same as they are in the linear theory and given by (4.1). From (4.2) we find

$$\begin{aligned} P_1(s_1, s_2) &= \Phi(s_2/\sqrt{\alpha_{00}}), \qquad & R_1(s_1, s_2) &= -\sqrt{\alpha_{00}}\,\Phi(s_2/\sqrt{\alpha_{00}}), \\ Q_1(s_1, s_2) &= 0, & S_1(s_1, s_2) &= 0. \end{aligned} \tag{5.2}$$

X_1 and t_1 are given exactly as in (4.3). From (4.5)–(4.7) we find

$$\begin{aligned} P_2(s_1, s_2) &= 0, \qquad & R_2(s_1, s_2) &= -\tfrac{1}{2}a_{41}\Phi^2(s_2/\sqrt{\alpha_{00}}), \\ Q_2(s_1, s_2) &= 0, & S_2(s_1, s_2) &= 0, \end{aligned} \tag{5.3}$$

$$X_2(s_1, s_2) = c_1\Phi^2\left(\frac{s_2}{a_0}\right)\left(\frac{s_1}{a_0}-\frac{s_2}{a_0}\right)+\int_{s_1/a_0}^{s_2/a_0}\left[c_6\sqrt{\alpha_{00}}\Phi^2(\xi)+c_5\sqrt{\delta_{00}}\,\Phi\left(\frac{s_2}{a_0}\right)\Phi(\xi)\right]d\xi,$$

$$t_2(s_1, s_2) = \left[c_1\Phi^2\left(\frac{s_2}{a_0}\right)\right]\left(\frac{s_1}{a_0}-\frac{s_2}{a_0}\right)+\int_{s_1/a_0}^{s_2/a_0}\left[c_6\Phi^2(\xi)+c_5\Phi\left(\frac{s_2}{a_0}\right)\Phi(\xi)\right]d\xi.$$

So, using (3.1), we get

$$\begin{aligned} P &= P_0\Phi(s_2\sqrt{\alpha_{00}})+O(P_0{}^3), \qquad Q = O(P_0{}^3), \\ R &= -\sqrt{\alpha_{00}}P_0\Phi(s_2/\sqrt{\alpha_{00}})-\tfrac{1}{2}a_{41}P_0{}^2\Phi^2(s_2/\sqrt{\alpha_{00}})+O(P_0{}^3), \\ S &= O(P_0{}^3), \\ X &= X_0+P_0X_1+P_0{}^2X_2+O(P_0{}^3), \qquad t = t_0+P_0t_1+P_0{}^2t_2+O(P_0{}^3), \end{aligned} \tag{5.4}$$

where X_0 and t_0 are given by (4.1), X_1 and t_1 by (4.3), and X_2 and t_2 by (5.3).

To further specialize this example we consider the case

$$\Phi(t) = \frac{t}{\tau}H(t)-\left(\frac{t}{\tau}-1\right)H(t-\tau), \tag{5.5}$$

Where

$$H(t) = \begin{cases} 0, t \le 0 \\ 1, t > 0 \end{cases}$$

is the unit step function. The function Φ is depicted graphically as Fig. 2.

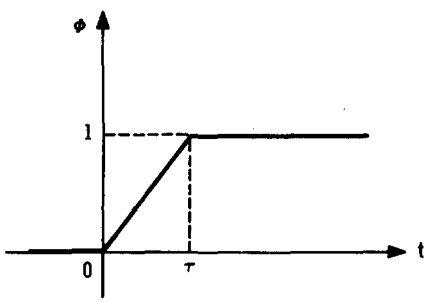

FIG. 2

In order to evaluate (5.4) it is convenient to divide the (X, t)-plane into the six regions shown in Fig. 3.

The interiors of these regions are characterizied by the inequalities

$$\begin{aligned} &\mathrm{I}: \sqrt{\alpha_{00}}\,\tau < s_1, && \sqrt{\alpha_{00}}\,\tau < s_2 \\ &\mathrm{II}: 0 < s_1 < \sqrt{\alpha_{00}}\,\tau, && \sqrt{\alpha_{00}}\,\tau < s_2 \\ &\mathrm{III}: s_1 < 0, && \sqrt{\alpha_{00}}\,\tau < s_2 \\ &\mathrm{IV}: s_1 < 0, && 0 < s_2 < \sqrt{\alpha_{00}}\,\tau \\ &\mathrm{V}: s_1 < 0, && s_2 < 0 \\ &\mathrm{VI}: 0 < s_1 < \sqrt{\alpha_{00}}\,\tau, && 0 < s_2 < \sqrt{\alpha_{00}}\,\tau \end{aligned} \tag{5.6}$$

In general we expect Region IV to be occupied by the longitudinal wave and Region II to be occupied by the shear wave. Regions I, III, and V are regions of uniform motion, Region V being the rest zone ahead of the disturbance and Region I the region of uniform motion behind the disturbance.

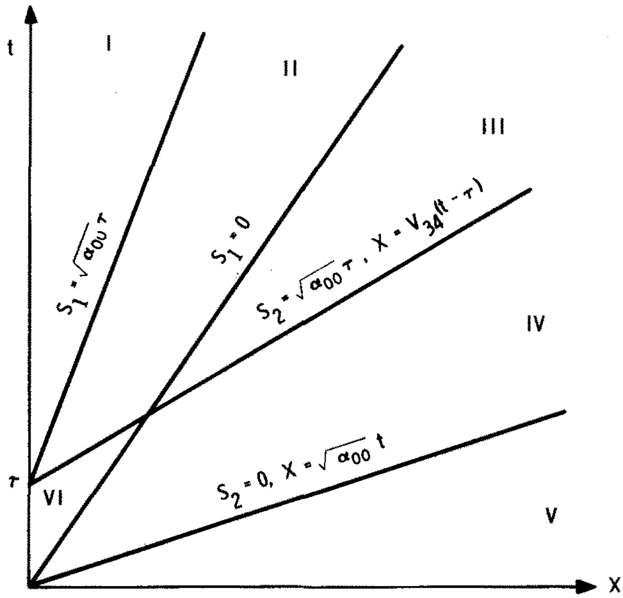

FIG. 3

In the present example we expect the shear wave to be absent so that the solution in Regions I, II, and III and in Regions IV and VI will be the same. This turns out to be the case for P, Q, R, and S, but the functions $X(s_1, s_2)$ and $t(s_1, s_2)$ are different in each of the six regions. However, the wavespeeds calculated from them by means of (2.19) conform to the above expectations.

We find the solution to be

Regions I, II, and III:

$$
\begin{aligned}
&P = P_0 + O(P_0{}^3), \qquad R = -\sqrt{\alpha_{00}}\,P_0 - \tfrac{1}{2}a_{41}P_0{}^2 + O(P_0{}^3),\\
&a_3 = \frac{X_{s_2}}{t_{s_2}} = \sqrt{\delta_{00} + a_{31}P_0 + O(P_0{}^2)}, \qquad a_4 = \frac{X_{s_1}}{t_{s_1}} = \sqrt{\alpha_{00} + a_{41}P_0 + (P_0{}^2)},
\end{aligned}
\tag{5.7}
$$

Regions IV and VI:

$$
\begin{aligned}
&P = P_0\frac{s_2}{\sqrt{\alpha_{00}}\,\tau} + O(P_0{}^3), \qquad R = -P_0\frac{s_2}{\tau} - \tfrac{1}{2}a_{41}P_0{}^2\left(\frac{s_2}{\sqrt{\alpha_{00}}\,\tau}\right)^2 + O(P_0{}^3),\\
&a_3 = \sqrt{\delta_{00} + a_{31}P_0\frac{s_2}{\sqrt{\alpha_{00}}\,\tau} + O(P_0{}^2)}, \qquad a_4 = \sqrt{\alpha_{00} + a_{41}P_0\frac{s_2}{\sqrt{\alpha_{00}}\,\tau} + O(P_0{}^2)}.
\end{aligned}
\tag{5.8}
$$

Region V:

$$
\begin{aligned}
&P = O(P_0{}^3), \qquad R = O(P_0{}^3),\\
&a_3 = \sqrt{\delta_{00} + O(P_0{}^2)}, \qquad a_4 = \sqrt{\delta_{00} + O(P_0{}^2)}.
\end{aligned}
\tag{5.9}
$$

From $(5.8)_4$ we find that the boundaries of Region IV move with the (material) velocities

$$
V_{34} = \sqrt{\alpha_{00} + a_{41}P_0 + O(P_0{}^2)}, \qquad V_{45} = \sqrt{\alpha_{00} + O(P_0{}^2)}. \tag{5.10}
$$

One sees immediately that the solutions, as given, satisfy the boundary and initial conditions to the order shown. Note also that the solutions are continuous across the leading and trailing edges of the wave ($s_2 = 0$ and $s_2 = \sqrt{\alpha_{00}}\,\tau$, respectively) as is appropriate for acceleration waves.

For the wave to be of increasing thickness, as it is shown in Fig. 3, a_4 must be a monotone decreasing function of s_2 on the interval $0 < s_2 < \sqrt{\alpha_{00}}\,\tau$. As a first approximation this is accomplished if $a_{41}P_0 < 0$. In the case that a_4 is an increasing function of s_2 on the interval in question the wave will coalesce into a shock after some finite time.

Examination of (5.7)–(5.10) reveals that P, Q, and the longitudinal wavespeed a_4 depend only on the coefficients of $\alpha(P, 0)$. This is to be expected since, as seen from (2.7), (2.13), and (2.17), the longitudinal wave problem is set in terms of the equations and boundary conditions

$$\alpha(P)P_X = R_t, \qquad R_X = P_t, \qquad P(X,0) = 0, \qquad R(X,0) = 0, \qquad P(0,t) = \varepsilon\Phi(t).$$

which involves only the coefficient α. The occurrence of other coefficients in the expressions for X and t is a consequence of the use of the shear wave characteristics as coordinates.

It is clear that if only longitudinal waves are to be considered one should recast the whole solution so as to use the advancing and receding characteristics for longitudinal disturbances for coordinates, as is the custom in gasdynamical calculations. We have chosen the present course in order to be able to discuss shear waves, as is done in the next example.

6. EXAMPLE 2: SHEAR WAVE

In this section we consider the shear loading problem. For this case we take

$$\Phi = 0, \qquad \varepsilon = Q_0, \qquad a_0 = \sqrt{\delta_{00}}, \qquad \xi = \frac{s_2}{\tau}\sqrt{\frac{\rho_0}{\mu}}, \qquad \eta = \frac{s_1}{\tau}\sqrt{\frac{\rho_0}{\mu}}. \tag{6.1}$$

To obtain the solution for this problem we need only substitute (6.1) into results obtained in Section 4. In order to make the example more explicit we choose

$$\Psi(t) = \frac{t}{\tau}H(t) - \left(\frac{t}{\tau} - 1\right)H(t-\tau), \tag{6.2}$$

where τ is a given constant. Associated with this problem we have the (X, t)-diagram of Fig. 4.

In Region IV of the figure the results given in Section 4 take the form

$$\begin{aligned} &P_1 = 0, \quad R_1 = 0, \quad Q_1 = 0, \quad S_1 = 0, \quad X_1 = 0, \quad t_1 = 0, \\ &P_2 = \frac{f_7}{f_8}\xi^2, \quad R_2 = -\sqrt{\alpha_{00}}P_2, \quad Q_2 = 0, \quad S_2 = 0, \\ &\frac{t_2}{\tau} = [c_3(\eta-\xi) + \tfrac{1}{3}c_4\xi]\xi^2, \quad \frac{X_2}{\tau}\sqrt{\frac{\rho_0}{\mu}} = \frac{t_2}{\tau} + \frac{1}{3}\frac{f_4 f_7}{f_8\sqrt{\delta_{00}}}\xi^3 \end{aligned} \tag{6.3}$$

From this we find the (material) velocity of the common boundary of Regions III and IV to be

$$V_{34} = \sqrt{\alpha_{00}} + a_{41}\frac{f_7}{f_8}{Q_0}^2 + \dots \tag{6.4}$$

A similar calculation for Region II of the figure gives

$$
\begin{aligned}
&P_1 = 0, \quad R_1 = 0, \quad Q_1 = \eta, \quad S_1 = -\eta\sqrt{(\mu/\rho_0)}, \quad X_1 = 0, \quad t_1 = 0,\\
&P_2 = -\frac{f_7}{f_8}(\eta^2-1), \quad R_2 = \frac{f_7}{f_8}(\sqrt{\delta_{00}}\,\eta^2-\sqrt{\alpha_{00}}), \quad Q_2 = 0, \quad S_2 = 0,\\
&\frac{t_2}{\tau} = (c_3+c_2\eta^2)(\eta-\xi)-\tfrac{1}{3}c_4(\eta^3-3\xi+2),\\
&\frac{X_2}{\tau}\sqrt{\frac{\rho_0}{\mu}} = (\sqrt{\delta_{00}}\,c_3+\sqrt{\alpha_{00}}\,c_2\eta^2)(\eta-\xi)-\tfrac{1}{3}c_8(\eta^3-3\xi+2).
\end{aligned}
\tag{6.5}
$$

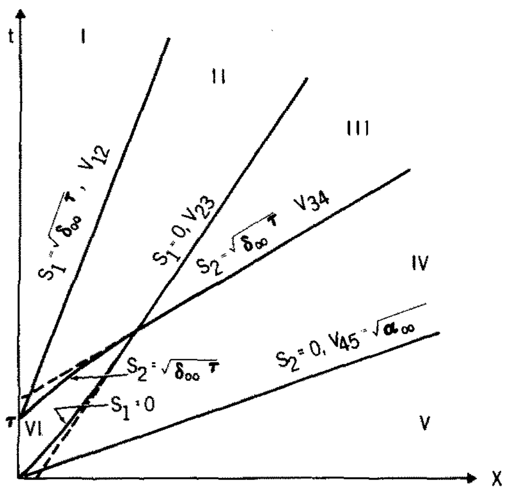

FIG. 4

And the velocities of the boundaries of this region are found to be

$$
V_{12} = \sqrt{\delta_{00}}+a_{32}{Q_0}^2+\dots, \qquad V_{23} = \sqrt{\delta_{00}}+a_{31}\frac{f_7}{f_8}{Q_0}^2+\dots \tag{6.6}
$$

The solution in Region V is just the rest solution given by the initial conditions. The solution in Region I is the final uniform motion and is obtained by evaluating the solution of Region II for $s_1 = a_0\tau$. In Region III the solution is one of uniform motion which agrees with the solutions in Regions II and IV evaluated on their boundaries with this region. The solution in and the boundaries of Region VI are readily calculated in the same manner as the solution in the other regions. They are more complicated because of the effects of interaction of the two waves.

From (6.3) we see that the longitudinal wave present in Region IV is of second order in Q_0. We already knew, of course, that it would have to be of at least second order since it does not occur in the linear theory.

7. EXAMPLE

As an example problem we further evaluate the results of Section 6 for the case

$$t_{ij} = \mu\left(\delta_{ij} - \frac{c_{ij}}{\sqrt{\mathrm{III}_{c^{-1}}}}\right). \tag{7.1}$$

This constitutive equation has been proposed by Ko[10] as a description of a 47% by volume polyurethane foam rubber having a shear modulus of 32 lb/in^2. In this case we have

$$\sigma(P, Q) = \mu\left(1 - \frac{1+Q^2}{(1+P)^3}\right), \qquad \tau(P, Q) = \mu\frac{Q}{(1+P)^2}. \tag{7.2}$$

Expansion about $P = Q = 0$ and comparison with (3.2) gives

$$\begin{aligned} &\rho_0\alpha_{00} = 3\mu, && \rho_0\alpha_{10} = -12\mu, && \rho_0\beta_{01} = -2\mu, \\ &\rho_0\alpha_{02} = 3\mu, && \rho_0\alpha_{20} = 30\mu, && \rho_0\delta_{00} = \mu, \\ &\rho_0\delta_{10} = -2\mu, && \rho_0\delta_{20} = 3\mu, && \rho_0\delta_{02} = 0. \end{aligned} \tag{7.3}$$

and hence we find that

$$a_{31} = a_{32} = -\sqrt{(\mu/\rho_0)}, \qquad a_{41} = -2\sqrt{3}\sqrt{(\mu/\rho_0)}, \qquad a_{42} = \frac{5}{2\sqrt{3}}\sqrt{\frac{\mu}{\rho_0}}, \tag{7.4}$$

and that

$$\begin{aligned} &c_2 = 4{\cdot}847, \qquad c_3 = 3{\cdot}232, \qquad c_4 = 2{\cdot}155, \\ &c_8 = 3{\cdot}338\sqrt{\frac{\mu}{\rho_0}}, \qquad \frac{f_4 f_7}{f_8} = 1{\cdot}183\sqrt{\frac{\mu}{\rho_0}}. \end{aligned} \tag{7.5}$$

As in the previous section, it is convenient to write

$$\xi = \frac{s_2}{\tau}\sqrt{\frac{\rho_0}{\mu}}, \qquad \eta = \frac{s_1}{\tau}\sqrt{\frac{\rho_0}{\mu}}. \tag{7.6}$$

In this notation (4.1) becomes

$$\frac{X_0}{\tau}\sqrt{\frac{\rho_0}{\mu}} = 2{\cdot}366(\xi - \eta), \qquad \frac{t_0}{\tau} = 2{\cdot}366\xi - 1{\cdot}366\eta. \tag{7.7}$$

In Region IV of the (X, t)-plane we have

$$\begin{aligned} &P_1 = Q_1 = 0, \qquad R_1 = S_1 = 0, \qquad X_1 = 0, \qquad t_1 = 0, \\ &P_2 = -0{\cdot}500\xi^2, \qquad R_2\sqrt{\frac{\rho_0}{\mu}} = 0{\cdot}866\xi^2, \qquad Q_2 = 0, \qquad S_2 = 0, \\ &t_2 = (3{\cdot}232\eta - 2{\cdot}513\xi)\xi^2, \qquad \frac{X_2}{\tau}\sqrt{\frac{\rho_0}{\mu}} = (3{\cdot}232\eta - 2{\cdot}118\xi)\xi^2. \end{aligned} \tag{7.8}$$

From (4.13) and (4.14) we find that

$$\frac{U_1}{\tau}\sqrt{\frac{\rho_0}{\mu}} = 0{\cdot}289{Q_0}^2\xi^3, \qquad \frac{U_2}{\tau}\sqrt{\frac{\rho_0}{\mu}} = 0. \tag{7.9}$$

In Region II of the (X, t)- plane we have

$$\begin{aligned} &P_1 = 0, \quad Q_1 = \eta, \quad R = 0, \quad S_1\sqrt{\frac{\rho_0}{\mu}} = -\eta, \quad X_1 = 0, \quad t_1 = 0, \\ &P_2 = 0{\cdot}500(\eta^2-1), \quad Q_2 = 0, \quad R_2\sqrt{\frac{\rho_0}{\mu}} = -0{\cdot}500(\eta^2-1{\cdot}732), \quad S_2 = 0, \\ &\frac{t_2}{\tau} = (3{\cdot}232+4{\cdot}847\eta^2)(\eta-\xi)-0{\cdot}718(\eta^3-3\xi+2). \end{aligned} \tag{7.10}$$

$$\frac{X_2}{\tau}\sqrt{\frac{\rho_0}{\mu}} = (0{\cdot}106-8{\cdot}395\eta^2)\xi+7{\cdot}282\eta^3+3{\cdot}232\eta-2{\cdot}225$$

As before we compute **U**, finding

$$\begin{aligned} \frac{U_1}{\tau}\sqrt{\frac{\rho_0}{\mu}} &= {Q_0}^2[0{\cdot}866\xi-0{\cdot}167\eta^3-0{\cdot}577], \\ \frac{U_2}{\tau}\sqrt{\frac{\rho_0}{\mu}} &= -\tfrac{1}{2}Q_0\eta^2. \end{aligned} \tag{7.11}$$

Finally, in Region I we have

$$\begin{aligned} &P_1 = 0, \quad Q_1 = 1, \quad R_1 = 0, \quad S_1\sqrt{\frac{\rho_0}{\mu}} = -1, \quad X_1 = 0, \quad t_1 = 0, \\ &P_2 = Q_2 = 0, \quad R_2 = 0{\cdot}366\sqrt{\frac{\mu}{\rho_0}}, \quad S_2 = 0, \\ &\frac{t_2}{\tau} = -5{\cdot}924(\xi-\eta), \quad \frac{X_2}{\tau}\sqrt{\frac{\rho_0}{\mu}} = -8{\cdot}287(\xi-\eta) \end{aligned} \tag{7.12}$$

$$\frac{U_1}{\tau}\sqrt{\frac{\rho_0}{\mu}} = {Q_0}^2(0{\cdot}866\xi-0{\cdot}500\eta-0{\cdot}244), \qquad \frac{U_2}{\tau}\sqrt{\frac{\rho_0}{\mu}} = -Q_0(\eta-0{\cdot}500).$$

In Region III the solution P, Q, R, S is just the constant matching Regions II and IV, and **U** is the straight line matching these regions.

Using these results we can plot curves showing the waveform from various points of view. In Figs. 5 and 6 the two dimensionless spatial displacement components $u_1/\tau \,.\, \sqrt{\rho_0/\mu}$ and $u_2/\tau \,.\, \sqrt{\rho_0/\mu}$ are displayed as functions of the dimensionless spatial (laboratory) co-ordinate for a sequence of times. This presentation of information is similar to a sequence of photographs of the waveform. In Figs. 7 and 8 the displacement components are displayed as functions of the dimensionless time t/τ for various observation points $x/\tau \,.\, \sqrt{\rho_0/\mu}$. This presentation of information is similar to that which would be recorded by an observer situated at a fixed place in the laboratory and watching the wave as it passes his station. There are other possibilities which are also worthy of consideration, for example a plot of

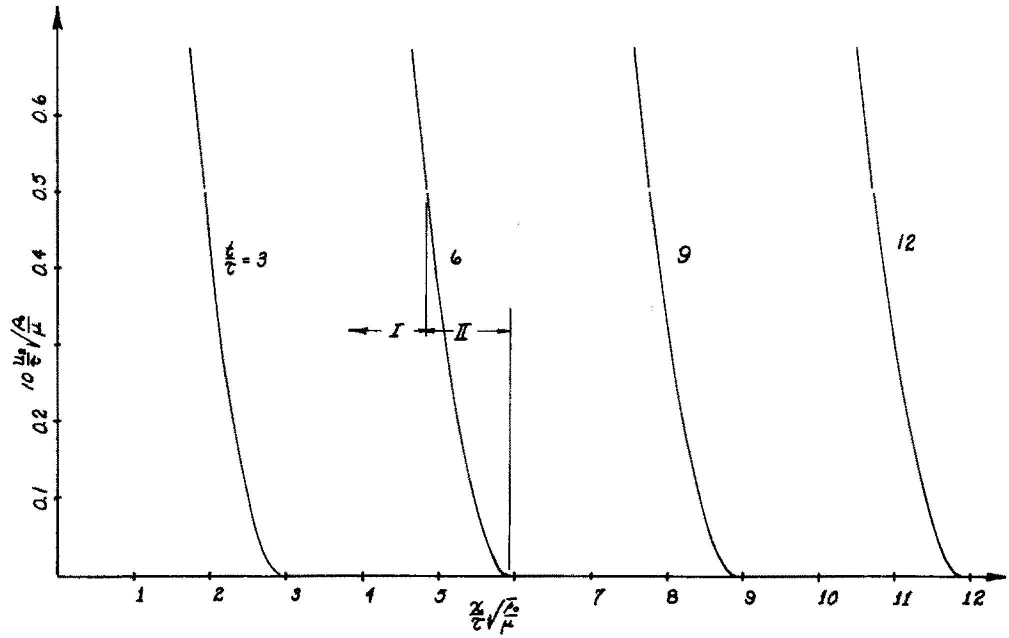

FIG. 5

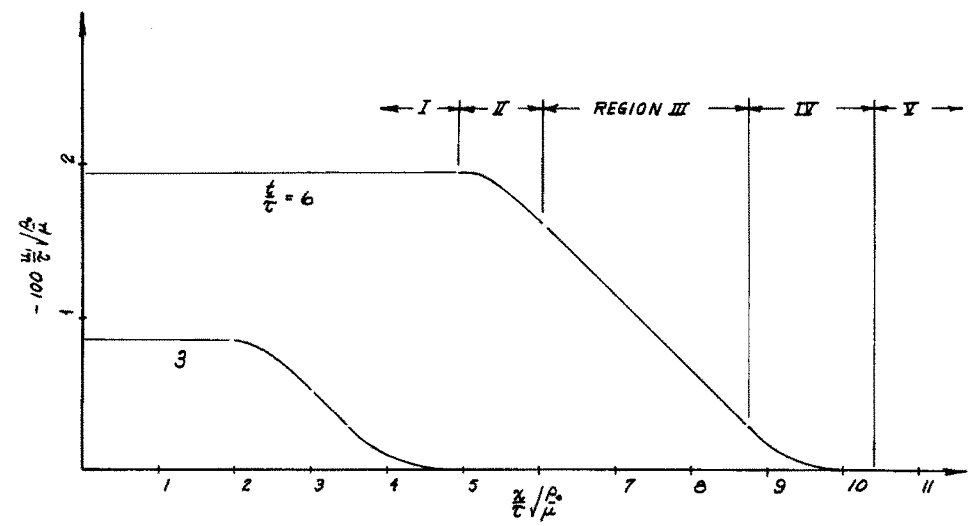

FIG. 6

displacement against time as apparent to an observer situated on a (displaced and moving) particle. This is the sort of information that would be generated by an instrument mounted on the specimen instead of being fixed in the laboratory. In the present example the difference between the material and spatial points of view does not show up well since the longitudinal wave, being of second order, involves only rather small displacements.

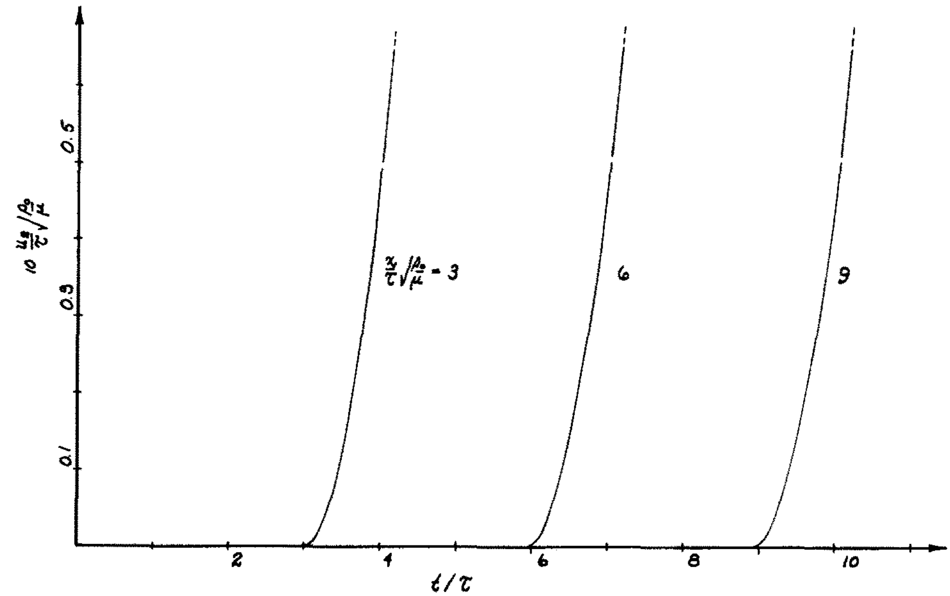

FIG. 7

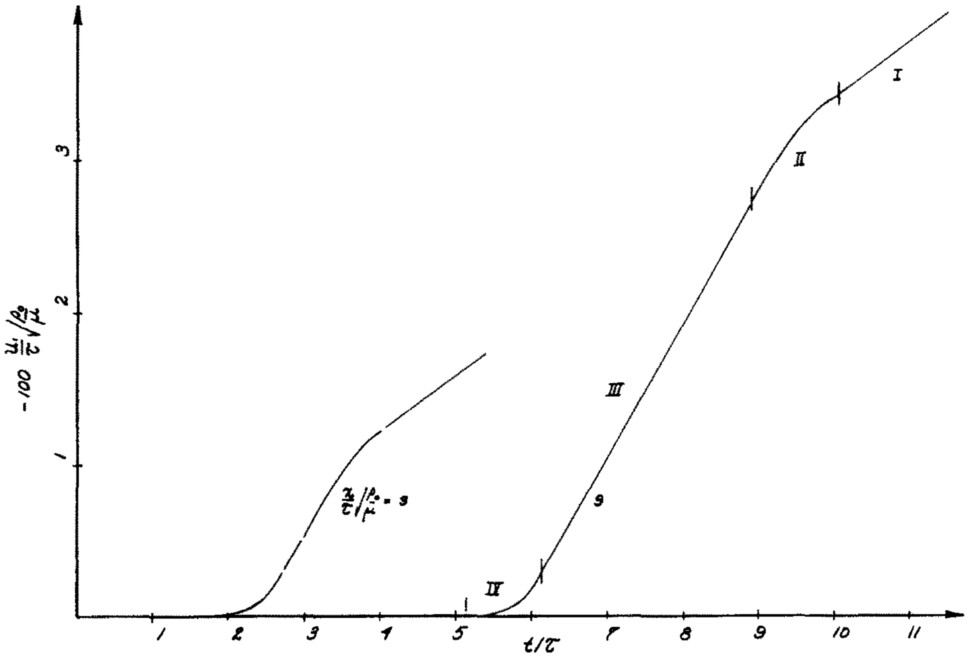

FIG. 8

Examination of the longitudinal displacement shown in Figs. 6 and 8 reveals that a shock is forming in the part of the curve corresponding to Region IV of Fig. 4, i.e. for this particular material the boundaries of the longitudinal wave region are converging instead of separating as shown there. This part of the disturbance is shown in Fig. 9 for larger values of

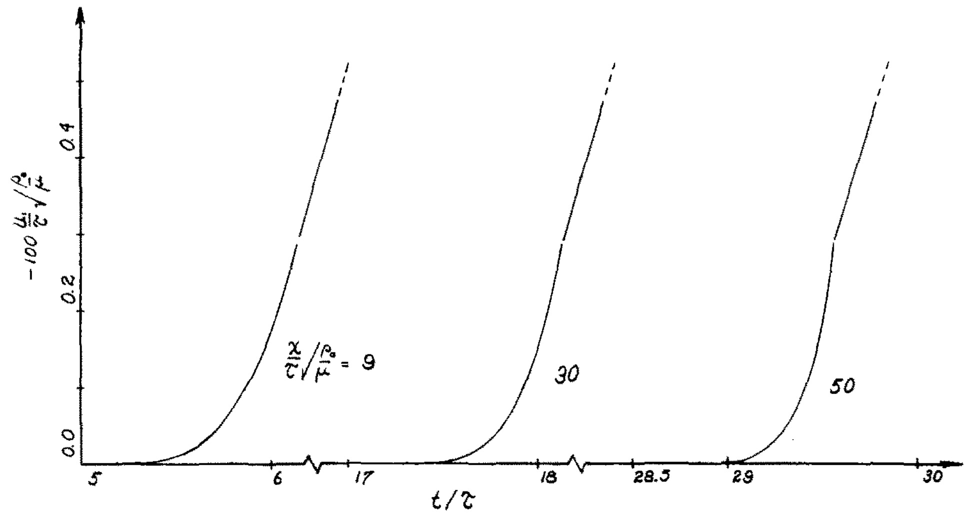

FIG. 9

$x/\tau \,.\, \sqrt{(\rho_0/\mu)}$, that is, for cases where the wave has propagated farther from the boundary of the domain. In this figure the tendency toward shock formation is becoming more apparent. That the shock actually forms and is not just a spurious result of the perturbation theory is proved in [1]. It is also shown there, consistent with the present calculation, that the shear wave becomes smoother rather than forming a shock. If we regard shock formation to have occurred when the width of Region IV becomes zero then we have the time of shock formation as that time when

$$\frac{x}{\tau}\sqrt{\frac{\rho_0}{\mu}}\bigg|_{\mathrm{IV},\xi=0} \quad \frac{x}{\tau}\sqrt{\frac{\rho_0}{\mu}}\bigg|_{\mathrm{IV},\xi=1},$$

that is, when

$$\frac{t}{\tau} = \frac{1}{Q_0{}^2} + 0{\cdot}909 + O(Q_0). \tag{7.13}$$

The place where the shock forms is that where

$$t/\tau|_{\mathrm{IV},\xi=0} = t/\tau|_{\mathrm{IV},\xi=1},$$

that is, when

$$\frac{x}{\tau}\sqrt{\frac{\rho_0}{\mu}} = 1{\cdot}732\frac{1}{Q_0{}^2} - 0{\cdot}159 + O(Q_0). \tag{7.14}$$

For $Q_0 = 0{\cdot}1$ shock formation occurs at approximately

$$\frac{t}{\tau} = 101, \qquad \frac{x}{\tau}\sqrt{\frac{\rho_0}{\mu}} = 173.$$

We see that the time and place of formation of the shock depends in a very sensitive way on the strength of the disturbance, which is measured by Q_0.

The general shape of curves such as those shown in Figs. 5–9 does not change very much with Q_0; it simply takes longer for a given shape to evolve as Q_0 becomes small. Although it does not affect the determination of the wave shape, we remark that observations taken at a given place cannot be continued for arbitrarily long times; the boundary moves past an observer stationed at $x/\tau \, . \, \sqrt{(\rho_0/\mu)}$ for

$$\frac{t}{\tau} = \frac{1}{Q_0^{\,2}} 2{\cdot}732 \frac{x}{\tau}\sqrt{\frac{\rho_0}{\mu}} + 0{\cdot}667 + O(Q_0). \tag{7.15}$$

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Абстракт—В этой статье исследуется метод эозмущений для получения одинаковых приближений при решений задач по распространению плоской волны в конечной теории упругости. Даются примеры, которые показывают применение этого метода к задачам по распространению продольной волны и волны сдвига. Дается численный пример, касающийся распространения волны сдвига в пенистой резине из полиуретана.